OKOUNKOV BODIES AND SESHADRI CONSTANTS

ATSUSHI ITO

ABSTRACT. Okounkov bodies, which are closed convex sets defined for big line bundles, have rich information of the line bundles. On the other hand, Seshadri constants are invariants which measure the positivity of line bundles. In this paper, we prove that Okounkov bodies give lower bounds of Seshadri constants. We also investigate Seshadri constants in toric cases.

CONTENTS

1. INTRODUCTION

In [LM], Lazarsfeld and Mustaţă defined the Okounkov body $\Delta(L)$ for a big line bundle L (or more generally, for a graded linear series). Since $\Delta(L)$ captures much of the asymptotic behaviors of the linear series $|kL|$ for $k \to +\infty$, $\Delta(L)$ is useful for studying the geometry of L, like the volume of L. Hence, it is natural to consider which invariant of L is encoded in the Okounkov body $\Delta(L)$. The purpose of this paper is

2010 *Mathematics Subject Classification.* 14C20;14M25.
Key words and phrases. Okounkov body, Seshadri constant, graded linear series, toric variety.

to show that Okounkov bodies give lower bounds of Seshadri constants, which are invariants measuring positivity of line bundles.

Let $W_\bullet = \{W_k\}_k$ be a graded linear series associated to a line bundle L on an n-dimensional variety X over the complex number field $\mathbb{C}$. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$. If $W_\bullet$ is birational (see Definition 3.3 for the definition of "birational"), we can construct an n-dimensional closed convex set $\Delta(W_\bullet) = \Delta_{z,>}(W_\bullet)$ in $\mathbb{R}^n$, which is called the Okounkov body of $W_\bullet$ with respect to z and $>$. Okounkov bodies were introduced and studied by Lazarsfeld and Mustaţă [LM] and independently by Kaveh and Khovanskii [KK], based on the work of Okounkov [Ok1], [Ok2]. The Okounkov body $\Delta(W_\bullet)$ seems to have rich information of $W_\bullet$. For example the volume of $W_\bullet$ is $n!$ times the Euclidean volume of $\Delta(W_\bullet)$.

On the other hand, Demailly [De] defined an interesting invariant, Seshadri constant, which measures the local positivity of a nef line bundle at a point (or more generally, along a closed subscheme) on a projective variety. Seshadri constants relate to jet separations of adjoint bundles, Ross-Thomas' slope stabilities of polarized varieties [RT], Gromov width (an invariant in symplectic geometry) [MP], and so on. Since Seshadri constants have a lower semicontinuity, the Seshadri constant of a nef line bundle is constant for the very general choice of the point. Thus we can consider that Seshadri constants at very general points measure the global positivity of line bundles somehow. Nakamaye [Na] defined Seshadri constants at very general points for big line bundles. For a detailed treatment of Seshadri constants, we refer the reader to [La, Chapter 5] or [BDH+].

In this paper, we extend the notion of Seshadri constants slightly. Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a variety X, $r \in \mathbb{Z}_+ = \{x \in \mathbb{Z} \,|\, x > 0\}$, and $\overline{m} \in \mathbb{R}^r_+ = \{x \in \mathbb{R} \,|\, x > 0\}^r$. We can define an invariant $\varepsilon(X, W_\bullet; \overline{m}) \in \mathbb{R}_+ \cup \{+\infty\}$, which is called the Seshadri constant of $W_\bullet$ at very general points with weight $\overline{m}$.

Meanwhile, for any n-dimensional convex set $\Delta \subset \mathbb{R}^n$, we can define a birational monomial graded linear series $W_{\Delta,\bullet}$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$ on $(\mathbb{C}^\times)^n$. Thus we can define $s(\Delta; \overline{m}) := \varepsilon((\mathbb{C}^\times)^n, W_{\Delta,\bullet}; \overline{m})$ for $\overline{m} \in \mathbb{R}^r_+$. The main theorem of this paper states that Okounkov bodies give lower bounds of Seshadri constants:

Theorem 1.1 (=Theorem 5.7). *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on an n-dimensional variety X.*

Fix a local coordinate system $z = (z_1, \dots, z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$.

Then $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m}) (= \varepsilon((\mathbb{C}^\times)^n, W_{\Delta_{z,>}(W_\bullet),\bullet}; \overline{m}))$ holds for any $r \in \mathbb{Z}_+$ and $\overline{m} \in \mathbb{R}^r_+$.

This theorem says that the Seshadri constant of a graded linear series $W_\bullet$ is greater than or equal to that of the monomial graded linear series $W_{\Delta(W_\bullet),\bullet}$ defined by the Okounkov body.

In view of Theorem 1.1, it is natural to investigate $s(\Delta; \overline{m})$. It is very difficult to compute $s(\Delta; \overline{m})$ in general, but when $r = 1$ and $\overline{m} = 1$ (in this case we write $s(\Delta)$ for short), we have the following theorem:

Theorem 1.2 (cf. Corollarys 6.3, 6.5)**.** *For an n-dimensional bounded convex set $\Delta \subset \mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}^n$, it holds that*

$$\begin{aligned} s(\Delta) &= \sup_{k \in \mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k} \\ &= \lim_{k \in \mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k}, \end{aligned}$$

where $A_{k\Delta,m}$ is a matrix which depends on $m, n \in \mathbb{N}$ and $k\Delta$.

As a special case, for an integral polytope $P \subset \mathbb{R}^n_{\geq 0}$ of dimension n,

$$\begin{aligned} \varepsilon(X_P, L_P; 1) &= \sup_{k \in \mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k} \\ &= \lim_{k \in \mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k} \end{aligned}$$

holds, where (X_P, L_P) is the polarized toric variety defined by P.

Since $s(\Delta)$ (or $s(P)$) does not change under parallel translations, the assumption that $\Delta \subset \mathbb{R}^n_{\geq 0}$ (or $P \subset \mathbb{R}^n_{\geq 0}$) is not essential. An important point is that $\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}$ can be computed if we know the lattice points in $k\Delta$. Therefore this theorem states that $s(\Delta)$ or the Seshadri constant $\varepsilon(X_P, L_P; 1)$ is the supremum (or the limit) of computable values. Thus we can obtain explicit lower bounds.

When $m > 1$, we state three methods to obtain lower bounds of $s(\Delta; \overline{m})$, though it is not enough for good estimations.

This paper is organized as follows: In Section 2, we prepare some notations and conventions. In Section 3, we define Seshadri constants for graded linear series. In Section 4, we introduce invariants for convex sets by using Seshadri constants. In Section 5, we give the proof of Theorem 1.1. In Section 6, we investigate the computations of invariants defined in Section 4.

Acknowledgments. The author wishes to express his gratitude to his supervisor Professor Yujiro Kawamata for his valuable advice, comments and warm encouragement. He is grateful to Professor Yasunari Nagai for many useful comments. He would like to express his gratitude to Professor Robert Lazarsfeld for valuable comments. He wishes to thank to Professor Marcin Dumnicki for answering his questions about references of results in Section 6. He would also like to thank Yoshinori Gongyo, Makoto Miura, Shinnosuke Okawa, and Yusaku Tiba for helpful discussions and comments.

The author is supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 23-56182) and the Grant-in-Aid for JSPS fellows.

2. Notations and conventions

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the set of all natural numbers, integers, rational numbers, real numbers, and complex numbers respectively. In this paper, $\mathbb{N}$ contains 0. Set $\mathbb{Z}_+ = \mathbb{N} \setminus 0, \mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$, and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$. For a real number $t \in \mathbb{R}$, the round up of t is denoted by $\lceil t \rceil \in \mathbb{Z}$.

For a subset $S \subset \mathbb{R}^n$, we denote by $\Sigma(S)$ the closed convex cone spanned by S. For $t \in \mathbb{R}_{\geq 0}$, we set $tS = \{tu \,|\, u \in S\}$. For another subset $S' \subset \mathbb{R}^n$, $S + S' = \{u + u' \,|\, u \in S, u' \in S'\}$ means the Minkowski sum of S and S'. For $u' \in \mathbb{R}^n$, $S + u' = \{u + u' \,|\, u \in S\}$ is the parallel translation of S by u'.

For a convex set $\Delta \subset \mathbb{R}^n$, we denote by $\mathrm{vol}(\Delta)$ the Euclidean volume of Δ in $\mathbb{R}^n$. The dimension of Δ is the dimension of the affine space spanned by Δ.

A subset $P \subset \mathbb{R}^n$ is called a polytope if it is the convex hull of a finite set in $\mathbb{R}^n$. A polytope P is integral (resp. rational) if all its vertices are contained in $\mathbb{Z}^n$ (resp. $\mathbb{Q}^n$).

For a subset S in a topological space, we denote by S° the interior of S.

For a variety X, we say a property holds at a general point of X if it holds for all points in the complement of a proper algebraic subset. A property holds at a very general point of X if it holds for all points in the complement of the union of countably many proper subvarieties.

Throughout this paper, a divisor means a Cartier divisor. Thus we use the words "divisor", "line bundle", and "invertible sheaf" interchangeably. We sometimes denote $L^{\otimes k}$ by kL for a line bundle L and an integer k. For divisors D and D', the inequality $D \geq D'$ means $D - D'$ is effective.

3. Seshadri constants of graded linear series

Throughout this paper, we consider varieties or schemes over the complex number field $\mathbb{C}$.

In this section, we define Seshadri constants for graded linear series.

Definition 3.1. Let L be a line bundle on a (not necessarily projective) variety X and W a subspace of $H^0(X, L)$. For $r \in \mathbb{Z}_+$ and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, we say that W separates $\overline{m}$-jets at smooth r points $p_1, \ldots, p_r$ in X if the natural map

$$W \to L/(L \otimes \bigotimes_{i=1}^{r} \mathfrak{m}_{p_i}^{m_i+1}) = \bigoplus_{i=1}^{r} L/\mathfrak{m}_{p_i}^{m_i+1} L$$

is surjective, where we regard $\mathfrak{m}_{p_i}^{m_i+1} = \mathcal{O}_X$ for $m_i \leq -1$. We say W generically separates $\overline{m}$-jets if W separates $\overline{m}$-jets at general r points in X. Note that any W generically separates $\overline{m}$-jets if $m_i \leq -1$ for any i.

We define $j(W; \overline{m}) \in \mathbb{R} \cup \{+\infty\}$ for $W \subset H^0(X, L)$ and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_+$ to be

$$j(W; \overline{m}) = \sup\{\, t \in \mathbb{R} \,|\, W \text{ generically separates } \lceil t\overline{m} \rceil\text{-jets}\},$$

where $\lceil t\overline{m} \rceil = (\lceil tm_1 \rceil, \ldots, \lceil tm_r \rceil)$. We denote it by $j(W)$ when $r = 1$ and $\overline{m} = 1 \in \mathbb{R}_+$. Note that

$$j(W; \overline{m}) = \max\{\, t \in \mathbb{R} \,|\, W \text{ generically separates } \lceil t\overline{m} \rceil\text{-jets}\} \in \mathbb{R}$$

when $\dim W < +\infty$.

When $W = H^0(X, L)$, we denote $j(W, \overline{m})$ by $j(L; \overline{m})$.

Remark 3.2. Note that W generically separates $\overline{m}$-jets if W separates $\overline{m}$-jets at *some* smooth r points $p_1, \ldots, p_r$ in X. We use this in Sections 5 and 6.

Suppose that X is a variety of dimension n and L is a line bundle on X. Let $W_\bullet = \{W_k\}_{k \in \mathbb{N}}$ be a graded linear series associated to L, i.e., W_k is a subspace of $H^0(X, kL)$ for any $k \geq 0$ with $W_0 = \mathbb{C}$, such that $\bigoplus_{k \geq 0} W_k$ is a graded subalgebra of the section ring $\bigoplus_{k \geq 0} H^0(X, kL)$. When all W_k are finite dimensional, $W_\bullet$ is called of finite dimensional type.

Definition 3.3. A graded linear series $W_\bullet$ on a variety X is birational if the function field $K(X)$ of X is generated by $\{\, f/g \in K(X) \,|\, f, g \in W_k, g \neq 0\}$ over $\mathbb{C}$ for any $k \gg 0$. When $W_\bullet$ is of finite dimensional type, this is clearly equivalent to the condition that the rational map defined by $|W_k|$ is birational onto its image for any $k \gg 0$, which is Condition (B) in [LM, Definition 2.5].

Now we define Seshadri constants for graded linear series:

Definition 3.4. Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a variety X.

For $\overline{m} = (m_1, \dots, m_r) \in \mathbb{R}_+^r$, we define the Seshadri constant of $W_\bullet$ at very general points with weight $\overline{m}$ to be

$$\varepsilon(X, W_\bullet; \overline{m}) = \varepsilon(W_\bullet; \overline{m}) := \sup_{k>0} \frac{j(W_k; \overline{m})}{k} \in \mathbb{R}_+ \cup \{+\infty\}.$$

Note that $j(W_k; \overline{m}) > 0$ holds for $k \gg 0$ by the birationality of $W_\bullet$.

When $W_k = H^0(X, kL)$ for any k, we denote it by $\varepsilon(X, L; \overline{m})$ or $\varepsilon(L; \overline{m})$ for short.

Remark 3.5. The definition of Seshadri constants by jet separations is due to Theorem 6.4 in [De]. See also [La] or [BDH+]. We treat the definition by blowing ups in Lemmas 3.11 and 3.12 later.

Remark 3.6. If subspaces W and W' in $H^0(X, L)$ satisfy the inclusion $W \subset W'$, then $j(W; \overline{m}) \leq j(W'; \overline{m})$ holds by definition. Thus $\varepsilon(W_\bullet; \overline{m}) \leq \varepsilon(W'_\bullet; \overline{m})$ holds for $W_\bullet, W'_\bullet$ associated to L if $W_k \subset W'_k$ for any k (we write $W_\bullet \subset W_\bullet{}'$ for such $W_\bullet, W'_\bullet$).

If $L \hookrightarrow L'$ is an injection between line bundles on X and W is a subspace of $H^0(X, L)$, then $j(W; \overline{m})$ does not change if we regard W as a subspace of $H^0(X, L')$ because we consider jets separations at general points. So $\varepsilon(W_\bullet; \overline{m})$ also dose not change for $W_\bullet$ which associated to L if we consider that $W_\bullet$ is associated to L'.

Let $\pi : X' \to X$ be a birational morphism and $W_\bullet$ a graded linear series associated to L on X. Then we can consider that $W_\bullet$ is a graded linear series on X' associated to $\pi^* L$ by the natural inclusion $H^0(X, kL) \subset H^0(X', k\pi^* L)$ for any $k \in \mathbb{N}$. By the similar reason as above, $\varepsilon(X, W_\bullet; \overline{m}) = \varepsilon(X', W_\bullet; \overline{m})$ holds.

By the following lemma, we may assume that $W_\bullet$ is of finite dimensional type in many cases when we prove properties of $\varepsilon(X, W_\bullet; \overline{m})$:

Lemma 3.7. *Let $W_\bullet$ be a birational graded linear series. Suppose that $W_{1,\bullet} \subset W_{2,\bullet} \subset \cdots \subset W_{l,\bullet} \subset \cdots \subset W_\bullet$ is an increasing sequence of birational graded linear series in $W_\bullet$ such that $W_k = \bigcup_{l=1}^{\infty} W_{l,k}$ holds for all k. Then $\varepsilon(W_\bullet; \overline{m}) = \sup_l \varepsilon(W_{l,\bullet}; \overline{m}) = \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ holds.*

Proof. The existence of $\lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ is clear because $\varepsilon(W_{l,\bullet}; \overline{m})$ is monotonically increasing. The inequality $\varepsilon(W_\bullet; \overline{m}) \geq \sup_l \varepsilon(W_{l,\bullet}; \overline{m}) \geq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ is also clear. Thus it is enough to show $\varepsilon(W_\bullet; \overline{m}) \leq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$.

Fix k and a real number $t < j(W_k;\overline{m})$. By definition, W_k generically separates $\lceil t\overline{m}\rceil$-jets. Hence $W_{l,k}$ also generically separates $\lceil t\overline{m}\rceil$-jets for $l \gg 0$ by the assumption $W_k = \bigcup_l W_{l,k}$. Thus it holds that $j(W_k;\overline{m})/k = \lim j(W_{l,k};\overline{m})/k \leq \lim_l \varepsilon(W_{l,\bullet};\overline{m})$. By definition of $\varepsilon(W_\bullet;\overline{m})$, we have $\varepsilon(W_\bullet;\overline{m}) \leq \lim_l \varepsilon(W_{l,\bullet};\overline{m})$. □

In Definition 3.4, $\varepsilon(W_\bullet;\overline{m})$ is defined by the supremum, but in fact it is the limit:

Lemma 3.8. *In Definition 3.4,* $\varepsilon(W_\bullet;\overline{m}) = \lim \dfrac{j(W_k;\overline{m})}{k}$ *holds.*

Proof. By Remark 3.7, we may assume that $W_\bullet$ is of finite dimensional type.

At first we show the following claim:

Claim 3.9. *Let $W_\bullet$ be a birational graded linear series of finite dimensional type associated to a line bundle L on a variety X. Then $j(W_{kl};\overline{m}) \geq l \cdot j(W_k;\overline{m})$ holds for any positive integer $k, l > 0$.*

Proof of Claim 3.9. For simplicity, we set $j_{k'} = j(W_{k'};\overline{m})$ in the proof of Claim 3.9. We prove this claim when $r = 1$. When $r > 1$, the proof is essentially same. So we leave the details to the reader.

We write $m \in \mathbb{R}_+$ instead of $\overline{m}$. Choose very general point $p \in X$ and set

$$W_{k',i} = W_{k'} \cap H^0(X, L^{\otimes k'} \otimes \mathfrak{m}_p^i) \subset H^0(X, L^{\otimes k'})$$

for $k', i \geq 0$. For any $j \in \mathbb{N}$, it is easy to show that,

$$W_{k'} \to L^{\otimes k'} \otimes \mathcal{O}_X/\mathfrak{m}_p^{j+1}$$

is surjective if and only if

$$W_{k',i} \to L^{\otimes k'} \otimes \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}$$

is surjective for each $i \in \{0, 1, \ldots, j\}$. Fix $i \in \{0, 1, 2, \ldots, l\lceil j_k m\rceil\}$. Then there exist integers $0 \leq i_1, \ldots, i_l \leq \lceil j_k m\rceil$ such that $i = i_1 + \ldots + i_l$. Consider the following diagram:

$$\begin{tikzcd}
W_{k,i_1} \otimes \cdots \otimes W_{k,i_l} \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & L^{\otimes k} \otimes \mathfrak{m}_p^{i_1}/\mathfrak{m}_p^{i_1+1} \otimes \cdots \otimes L^{\otimes k} \otimes \mathfrak{m}_p^{i_l}/\mathfrak{m}_p^{i_l+1} \arrow[d, "\beta"] \\
W_{kl,i} \arrow[r] & L^{\otimes kl} \otimes \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}.
\end{tikzcd}$$

In the above diagram, α is surjective because $i_1, \ldots, i_l \leq \lceil j_k m\rceil$, and β is clearly surjective. Hence the map

$$W_{kl,i} \to L^{\otimes kl} \otimes \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}$$

is also surjective for any $i \in \{0,1,2,\dots,l\lceil j_k m\rceil\}$. Thus W_{kl} generically separates $l\lceil j_k m\rceil$-jets, which means $j_{kl} \geq l\lceil j_k m\rceil/m \geq l \cdot j_k$. □

Now we return to the proof of the lemma. We only prove the case $\varepsilon(W_\bullet;\overline{m}) < +\infty$. When $\varepsilon(W_\bullet;\overline{m}) = +\infty$, the proof is similar.

By definition, it is sufficient to show

$$\liminf \frac{j(W_k;\overline{m})}{k} \geq \sup \frac{j(W_k;\overline{m})}{k}.$$

Fix $\delta > 0$ and choose k_0 such that $\dfrac{j(W_{k_0};\overline{m})}{k_0} > \varepsilon(W_\bullet;\overline{m}) - \delta$. Let N be a sufficiently large integer and choose a general $s_{N+i} \in W_{N+i}$ for each $i = 0,1,\dots,k_0-1$.

Fix $k \geq N$. Then k is written as $k = k_0 l + N + i$ for some natural numbers $l \in \mathbb{N}$ and $0 \leq i \leq k_0 - 1$. Then there is the injection

$$W_{k_0 l} \hookrightarrow W_k; s \mapsto s \cdot s_{N+i},$$

induced by the multiplication in $\bigoplus W_j$. Therefore it holds that

$$j(W_k;\overline{m}) \geq j(W_{k_0 l};\overline{m}) \geq l \cdot j(W_{k_0};\overline{m})$$

by Remark 3.6 and Claim 3.9. Since $l \to +\infty$ if $k \to +\infty$,

$$\liminf_k \frac{j(W_k;\overline{m})}{k} \geq \liminf_l \frac{l \cdot j(W_{k_0};\overline{m})}{k_0 l + N + i} \geq \frac{j(W_{k_0};\overline{m})}{k_0} > \varepsilon(W_\bullet;\overline{m}) - \delta.$$

By $\delta \to 0$, we finish the proof of Lemma 3.8. □

Many properties of Seshadri constants of ample line bundles also hold for graded linear series. We use the followings later:

Lemma 3.10. *Let $\overline{m}$ be in $\mathbb{R}^r_+$ and $W_\bullet$ (resp.$W'_\bullet$) a birational graded linear series associated to a line bundles L (resp. L') on a variety X. Then the followings hold:*

1. *$\varepsilon(W^{(l)}_\bullet;\overline{m}) = l \cdot \varepsilon(W_\bullet;\overline{m})$ holds for any positive integer $l \in \mathbb{Z}_+$, where $W^{(l)}_\bullet$ is the graded linear series associated to $L^{\otimes l}$ defined by $W^{(l)}_k = W_{kl}$.*
2. *$\varepsilon(W_\bullet;t\overline{m}) = t^{-1}\varepsilon(W_\bullet;\overline{m})$ holds for any positive real number $t > 0$.*
3. *$\varepsilon(W''_\bullet;\overline{m}) \geq \varepsilon(W_\bullet;\overline{m}) + \varepsilon(W'_\bullet;\overline{m})$ holds, where $W''_\bullet$ is the graded linear series associated to $L \otimes L'$ defined by $W''_k :=$ the image of $W_k \otimes W'_k \to H^0(X, L \otimes L')$.*
4. *$\varepsilon(W_\bullet;\overline{m}) \leq \sqrt[n]{\mathrm{vol}(W_\bullet)/|\overline{m}|_n}$ holds, where $\mathrm{vol}(W_\bullet) = \lim_k \dfrac{\dim W_k}{k^n/n!}$, $|\overline{m}|_n = \sum_{i=1}^r m_i^n$, and n is the dimension of X.*

Proof. By Lemma 3.8, we have

$$\begin{aligned}\varepsilon(W_\bullet^{(l)};\overline{m}) &= \lim_k \frac{j(W_k^{(l)};\overline{m})}{k} \\ &= \lim_k \frac{j(W_{kl};\overline{m})}{k} \\ &= l \cdot \lim_k \frac{j(W_{kl};\overline{m})}{kl} \\ &= l \cdot \lim_k \frac{j(W_k;\overline{m})}{k} = l \cdot \varepsilon(W_\bullet;\overline{m}).\end{aligned}$$

Hence (1) is shown.

We can show (2) easily from the definition and the following:

$$\begin{aligned}j(W_k;t\overline{m}) &= \sup\{\, s \in \mathbb{R} \mid W_k \text{ generically separates } \lceil st\overline{m}\rceil\text{-jets}\,\} \\ &= t^{-1}\sup\{\, s \in \mathbb{R} \mid W_k \text{ generically separates } \lceil s\overline{m}\rceil\text{-jets}\,\} \\ &= t^{-1} j(W_k;\overline{m}).\end{aligned}$$

To prove (3), it is sufficient to show that $j(W_k'';\overline{m}) \geq j(W_k;\overline{m}) + j(W_k';\overline{m})$ holds for any k. We can show this by the argument similar to the proof of Claim 3.9. We leave the details to the reader.

To show (4), we fix a positive number $0 < t < \varepsilon(W_\bullet;\overline{m})$. By Lemma 3.8, the equality $j(W_k;\overline{m}) > kt$ holds for any $k \gg 0$. Thus W_k separates $\lceil kt\overline{m}\rceil$-jets for any $k \gg 0$, i.e., the map

$$W_k \to \bigoplus_i L \otimes \mathcal{O}/\mathfrak{m}_{p_i}^{\lceil ktm_i\rceil+1}$$

is surjective for very general $p_1, \ldots, p_r$. This surjection induces the following inequality:

$$\begin{aligned}\dim W_k &\geq \dim \bigoplus_i L \otimes \mathcal{O}/\mathfrak{m}_{p_i}^{\lceil ktm_i\rceil+1} \\ &= \sum_i \binom{\lceil ktm_i\rceil + n}{n} = \sum_i \frac{t^n m_i^n}{n!} k^n + O(k^{n-1}).\end{aligned}$$

Thus we have $\mathrm{vol}(W_\bullet) \geq t^n |\overline{m}|_n$ by $k \to +\infty$. We finish the prove by $t \to \varepsilon(W_\bullet;\overline{m})$. $\square$

Note that Definition 3.4 is a natural generalization of the well known definition of the Seshadri constant (at very general points) for a nef and big line bundle, i.e.,

Lemma 3.11. *Let X be a projective variety, L a nef and big line bundle on X, and $\overline{m} = (m_1, \dots, m_r) \in \mathbb{R}^r_+$. Then it holds that*

$$\varepsilon(X, L; \overline{m}) = \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\},$$

where $\mu : \widetilde{X} \to X$ is the blowing up along very general r points in X and E_i are the exceptional divisors.

Proof. The proof is essentially same as that of [La, Theorem 5.1.17]. At first, we show $\varepsilon(X, L; \overline{m}) \leq \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$, i.e., $\mu^* L - \varepsilon(X, L; \overline{m}) \sum_{i=1}^{r} m_i E_i$ is nef. Fix a curve $C \subset X$ and let $\widetilde{C} \subset \widetilde{X}$ be the strict transform of C. It is enough to show $(\mu^* L - \varepsilon(X, L; \overline{m}) \sum_{i=1}^{r} m_i E_i).\widetilde{C} \geq 0$. For each k, the line bundle $L^{\otimes k}$ separates $\lceil j_k \overline{m} \rceil$-jets at very general points $p_1, \dots, p_r$, where $j_k := j(kL; \overline{m})$. Hence there exists a nonzero effective divisor $D \in |L^{\otimes k} \bigotimes_i \mathfrak{m}_{p_i}^{\lceil j_k m_i \rceil}|$ such that D does not contain C. Thus we have

$$\begin{aligned} \mu^* L.\widetilde{C} &= L.C \\ &= k^{-1} D.C \\ &\geq k^{-1} \sum_i \operatorname{mult}_{p_i}(D) \cdot \operatorname{mult}_{p_i}(C) \\ &\geq k^{-1} \sum_i \lceil j_k m_i \rceil \operatorname{mult}_{p_i}(C) \\ &\geq k^{-1} \sum_i j_k m_i \operatorname{mult}_{p_i}(C) \\ &= k^{-1} j_k \sum_i m_i E_i.\widetilde{C}, \end{aligned}$$

where $\operatorname{mult}_{p_i}$ is the multiplicity at p_i. By $k \to +\infty$, it follows that $\mu^* L.\widetilde{C} \geq \varepsilon(X, L; \overline{m}) \sum_{i=1}^{r} m_i E_i.\widetilde{C}$.

We show the opposite inequality. At first, we assume L is ample. Let $p_1, \dots, p_r$ be very general r points in X. Fix a rational number $0 < t = a/b < \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$ with positive integers a, b. Then $b\mu^* L - a \sum_{i=1}^{r} m_i E_i$ is an ample $\mathbb{R}$-line bundle on $\widetilde{X}$. Multiplying a and b by a sufficiently large positive integer, we may assume that $b\mu^* L - \sum_{i=1}^{r} \lceil a m_i \rceil E_i$ is an ample line bundle on $\widetilde{X}$. By Fujita's vanishing theorem (see [La, Theorem 1.4.35] for instance), there is a natural number N such that

$$H^1(\widetilde{X}, l(b\mu^* L - \sum_{i=1}^{r} \lceil a m_i \rceil E_i) + P) = 0$$

for every $l \geq N$ and every nef line bundle P. For any integer $k \geq Nb$, write $k = lb + b'$ with $0 \leq b' < b$. We can apply the above vanishing for $P = b'\mu^* L$, so we have

$$H^1(\widetilde{X}, k\mu^* L - l \sum_{i=1}^{r} \lceil am_i \rceil E_i) = 0.$$

Since $H^1(\widetilde{X}, k\mu^* L - l \sum_{i=1}^{r} \lceil am_i \rceil E_i) = H^1(X, kL \otimes \bigotimes_i \mathfrak{m}_{p_i}^{l\lceil am_i \rceil})$, the line bundle kL separates $(l\lceil am_1 \rceil - 1, \dots, l\lceil am_r \rceil - 1)$-jets at $p_1, \dots, p_r$. Hence

$$\frac{j(kL; \overline{m})}{k} \geq \min_i \frac{l\lceil am_i \rceil - 1}{km_i} = \min_i \frac{l\lceil am_i \rceil - 1}{(lb + b')m_i}.$$

By $k \to +\infty$, we have

$$\varepsilon(L; \overline{m}) \geq \lim_k \min_i \frac{l\lceil am_i \rceil - 1}{(lb + b')m_i} = a/b = t.$$

By $t \to \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$, it holds that $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$.

Next we show the nef and big case. Since L is nef and big, there exists an effective divisor E on X such that $L - sE$ is ample for any $0 < s \ll 1$. Fix a rational number $0 < s \ll 1$ and take a sufficiently divisible integer $l \in \mathbb{Z}_+$ such that $ls \in \mathbb{Z}$. Then $\varepsilon(l(L - sE); \overline{m}) \leq \varepsilon(lL; \overline{m}) = l \cdot \varepsilon(L; \overline{m})$ holds by Remark 3.6 and Lemma 3.10 (1). Since $l(L - sE)$ is ample, we have

$$\begin{aligned}
\varepsilon(l(L - sE); \overline{m}) &= \max\{\, t \geq 0 \,|\, \mu^*(l(L - sE)) - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\} \\
&= l \cdot \max\{\, t \geq 0 \,|\, \mu^*(L - sE) - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}.
\end{aligned}$$

Hence $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^*(L - sE) - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$ holds. By $s \to 0$, we have $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\}$. Thus we finish the proof. □

Note that $W_\bullet = \{H^0(X, kL)\}_k$ is birational if and only if L is big for a projective variety X. For a nef but not big line bundle L on X, we define $\varepsilon(X, L; \overline{m}) := \max\{\, t \geq 0 \,|\, \mu^* L - t \sum_{i=1}^{r} m_i E_i \text{ is nef}\,\} = 0$ according to nef and big cases.

For projective varieties, we can write Seshadri constants of graded linear series by using that of nef line bundles as follows:

Lemma 3.12. *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a projective variety X. For each $k > 0$, set*

$$M_k = \mu_k^* L - F_k,$$

where $\mu_k : X_k \to X$ is a resolution of the base ideal

$$\mathfrak{b}_k := \text{the image of } W_k \otimes L^{-k} \to \mathcal{O}_X$$

and $\mathcal{O}_{X_k}(-F_k) := \mu_k^{-1}\mathfrak{b}_k$. Set $M_k = 0$ if $W_k = 0$. Then it holds that

$$\varepsilon(X, W_\bullet; \overline{m}) = \sup_k \frac{\varepsilon(X_k, M_k; \overline{m})}{k} = \lim_k \frac{\varepsilon(X_k, M_k; \overline{m})}{k}.$$

Proof. Note that M_k is nef for any k. At first we show that $\lim_k \dfrac{\varepsilon(X_k, M_k; \overline{m})}{k}$ exists and the second equality in the above statement holds. To prove this, it suffices to show that

$$\frac{\varepsilon(X_{k_0}, M_{k_0}; \overline{m})}{k_0} \leq \liminf \frac{\varepsilon(X_k, M_k; \overline{m})}{k}$$

holds for any $k_0 > 0$. We may assume that M_{k_0} is big.

Fix a sufficiently large integer N. For each $k \geq N$, we can write $k = k_0 l + N + i$ for some $l \in \mathbb{N}$ and $0 \leq i \leq k_0 - 1$. Since $\varepsilon(X_{k'}, M_{k'}; \overline{m})$ does not depend on the resolution $\mu_{k'}$ by Remark 3.6 or Lemma 3.11, we may take a common resolution of $\mathfrak{b}_{k_0}, \mathfrak{b}_k$, and $\mathfrak{b}_{N+i}$. Then it is easy to see $M_k \geq lM_{k_0} + M_{N+i}$, in particular $M_k \geq lM_{k_0}$ holds because M_{N+i} is effective. By Remark 3.6 and Lemma 3.10 (1) or Lemma 3.11, it holds that

$$\varepsilon(M_k; \overline{m}) \geq \varepsilon(lM_{k_0}; \overline{m}) = l \cdot \varepsilon(M_{k_0}; \overline{m}).$$

The inequality $\dfrac{\varepsilon(X_{k_0}, M_{k_0}; \overline{m})}{k_0} \leq \liminf \dfrac{\varepsilon(X_k, M_k; \overline{m})}{k}$ follows from this immediately.

Next we show the first equality. Since $\mu_k^*|W_k| \subset |M_k|$, it holds that $j(W_k; \overline{m}) \leq j(M_k; \overline{m}) \leq \varepsilon(M_k; \overline{m})$ for any k. Thus we have

$$\varepsilon(X, W_\bullet; \overline{m}) = \lim \frac{j(W_k; \overline{m})}{k} \leq \lim \frac{\varepsilon(M_k; \overline{m})}{k}.$$

To show the opposite inequality, we use Lemma 3.11. Since $W_\bullet$ is birational, the morphism $\varphi_k : X_k \to \mathbb{P}^{\dim |W_k|}$ defined by $\mu_k^*|W_k|$ is birational onto its image for $k \gg 0$. Denote the image of φ_k by Y_k. By Lemma 3.11, $\varepsilon(X_k, M_k; \overline{m}) = \varepsilon(Y_k, \mathcal{O}_{Y_k}(1); \overline{m})$ holds because φ_k is birational. Furthermore $W_k^l = H^0(Y_k, \mathcal{O}_{Y_k}(l))$ for $l \gg 0$, where W_k^l is the image of $W_k^{\otimes l} \to W_{kl}$. This implies that $j(\mathcal{O}_{Y_k}(l); \overline{m}) = j(W_k^l; \overline{m}) \leq$

$j(W_{kl};\overline{m})$ for $l \gg 0$. Thus we have

$$\begin{aligned}\frac{\varepsilon(X_k, M_k;\overline{m})}{k} &= \frac{\varepsilon(Y_k, \mathcal{O}_{Y_k}(1);\overline{m})}{k} \\ &= \frac{1}{k}\lim_{l}\frac{j(\mathcal{O}_{Y_k}(l);\overline{m})}{l} \\ &\leq \lim \frac{j(W_{kl};\overline{m})}{kl} = \varepsilon(W_{\bullet};\overline{m}).\end{aligned}$$

□

4. Monomial graded linear series on $(\mathbb{C}^\times)^n$

Let n be a positive integer. For a subset $S \subset \mathbb{R}^n$, we define

$$V_S := \bigoplus_{u \in S\cap\mathbb{Z}^n} \mathbb{C}x^u \subset \bigoplus_{u\in\mathbb{Z}^n}\mathbb{C}x^u = H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}).$$

For a convex set Δ in $\mathbb{R}^n$, we define a graded linear series $W_{\Delta,\bullet}$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$ by

$$W_{\Delta,k} := V_{k\Delta} \subset H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}) = H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}^{\otimes k}).$$

It is easy to see that $W_{\Delta,\bullet}$ is birational if and only if $\dim\Delta = n$. We investigate the Seshadri constant of $W_{\Delta,\bullet}$ in this section.

Definition 4.1. For a subset $S \subset \mathbb{R}^n$ and $\overline{m} \in \mathbb{R}^r_+$, we define $\tilde{s}(S;\overline{m})$ to be

$$\tilde{s}(S;\overline{m}) = j(V_S;\overline{m}) \in \mathbb{R}\cup\{+\infty\}.$$

For a convex set $\Delta \subset \mathbb{R}^n$, we define $s(\Delta;\overline{m})$ in $\mathbb{R}_{\geq 0}\cup\{+\infty\}$ to be

$$s(\Delta;\overline{m}) = \begin{cases} \varepsilon(W_{\Delta,\bullet};\overline{m}) & \text{if } \dim\Delta = n \\ 0 & \text{otherwise.}\end{cases}$$

When $r = 1$ and $\overline{m} = 1$, we denote them by $\tilde{s}(S)$ and $s(\Delta)$ instead.

Remark 4.2. For any Δ and $\overline{m}$, it holds that

$$s(\Delta;\overline{m}) = \sup_k \frac{\tilde{s}(k\Delta;\overline{m})}{k} = \lim_k \frac{\tilde{s}(k\Delta;\overline{m})}{k}.$$

When $\dim\Delta = n$, this follows from Lemma 3.8.

When $\dim\Delta < n$, $V_{k\Delta}$ does not generically separate 1-jets, which we will show later (see Proposition 6.1). Thus it holds that $\tilde{s}(k\Delta;\overline{m}) = j(V_{k\Delta};\overline{m}) \leq 0$. Moreover $V_{k\Delta}$ generically separates $(\underbrace{-1,\dots,-1}_{r})$-jets, hence $\tilde{s}(k\Delta;\overline{m}) = j(V_{k\Delta};\overline{m}) \geq -\max_i\{1/m_i\}$. Therefore we have $\sup_k \tilde{s}(k\Delta;\overline{m})/k = \lim_k \tilde{s}(k\Delta;\overline{m})/k = 0 = s(\Delta;\overline{m})$.

Remark 4.3. The following properties are easily shown from the definitions of $\tilde{s}(S;\overline{m})$ and $s(\Delta;\overline{m})$:

(1) $\tilde{s}(S_1;\overline{m}) \leq \tilde{s}(S_2;\overline{m}), s(\Delta_1;\overline{m}) \leq s(\Delta_2;\overline{m})$ for $S_1 \subset S_2$ and $\Delta_1 \subset \Delta_2$.
(2) $\tilde{s}(S+u;\overline{m}) = \tilde{s}(S;\overline{m})$ for $u \in \mathbb{Z}^n$.
(3) $s(t\Delta;\overline{m}) = t \cdot s(\Delta;\overline{m}), s(\Delta+u;\overline{m}) = s(\Delta;\overline{m})$ for any $t > 0 \in \mathbb{Q}$ and $u \in \mathbb{Q}^n$.

Proof. (1) is clear. (2) is immediately shown by the following diagram:

$$\begin{tikzcd} V_S \arrow[r, hook] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \arrow[d, "\wr"', "\times x^u"] \\ V_{S+u} \arrow[r, hook] & \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]. \end{tikzcd}$$

To show (3), choose sufficiently divisible $l \in \mathbb{Z}_+$ such that $lt \in \mathbb{Z}$ and $lu \in \mathbb{Z}^n$. Then

$$\begin{aligned} s(t\Delta;\overline{m}) &= \lim_k \frac{\tilde{s}(kt\Delta;\overline{m})}{k} \\ &= \lim_k \frac{\tilde{s}(klt\Delta;\overline{m})}{kl} \\ &= t \lim_k \frac{\tilde{s}(klt\Delta;\overline{m})}{klt} = t \cdot s(\Delta;\overline{m}) \end{aligned}$$

and by (2),

$$\begin{aligned} s(\Delta+u;\overline{m}) &= \lim_k \frac{\tilde{s}(k(\Delta+u);\overline{m})}{k} \\ &= \lim_k \frac{\tilde{s}(kl(\Delta+u);\overline{m})}{kl} \\ &= \lim_k \frac{\tilde{s}(kl\Delta+klu;\overline{m})}{kl} \\ &= \lim_k \frac{\tilde{s}(kl\Delta;\overline{m})}{kl} = s(\Delta;\overline{m}). \end{aligned}$$

□

Following lemmas are properties of $s(\Delta;\overline{m})$ which are used later.

Lemma 4.4. *Let Δ be a convex set in $\mathbb{R}^n$. Then $s(\Delta;\overline{m}) = s(\Delta^\circ;\overline{m})$.*

Proof. If $\dim\Delta < n$, then $s(\Delta;\overline{m}) = s(\Delta^\circ;\overline{m}) = 0$ by definition.

So we may assume $\dim\Delta = n$ and it is enough to show $s(\Delta;\overline{m}) \leq s(\Delta^\circ;\overline{m})$. Fix $u \in \Delta^\circ \cap \mathbb{Q}^n$. By the convexity of Δ, there exists the

inclusion $\Delta - u \subset t(\Delta^\circ - u)$ for $t > 1$. Thus

$$s(\Delta;\overline{m}) = s(\Delta - u;\overline{m}) \leq s(t(\Delta^\circ - u);\overline{m}) = t \cdot s(\Delta^\circ;\overline{m})$$

holds for $t > 1$ in $\mathbb{Q}$ by Remark 4.3 (3). By $t \to 1$, we are done. □

Now we can show that the property (3) in Remark 4.3 holds for any $t \in \mathbb{R}_+$ and $u \in \mathbb{R}^n$:

Lemma 4.5. *Let $\Delta \subset \mathbb{R}^n$ be a convex set, $u \in \mathbb{R}^n$, and $t \in \mathbb{R}_+$. Then*

$$s(\Delta + u;\overline{m}) = s(\Delta;\overline{m}), \ \ s(t\Delta;\overline{m}) = t \cdot s(\Delta;\overline{m}).$$

Proof. These are all 0 if $\dim \Delta < n$, so we may assume $\dim \Delta = n$.

Fix $u' \in \Delta^\circ \cap \mathbb{Q}^n$. As in the proof of Lemma 4.4, $\Delta - u' \subset (1+t')(\Delta^\circ - u')$ holds for $t' > 0$. Translating it by $u + u'$, we have $\Delta + u \subset (1 + t')(\Delta^\circ - u') + u + u'$. Choose $u'' \in \mathbb{Q}^n$ such that $(u+u') - u'' \in t'(\Delta^\circ - u')$, then $\Delta + u \subset (1 + 2t')(\Delta^\circ - u') + u''$. Hence for $t' > 0$ in $\mathbb{Q}$,

$$s(\Delta + u;\overline{m}) \leq s((1 + 2t')(\Delta^\circ - u') + u'';\overline{m}) = (1 + 2t')s(\Delta;\overline{m})$$

holds by Remark 4.3 (3) and Lemma 4.4. Thus $s(\Delta + u;\overline{m}) \leq s(\Delta;\overline{m})$ follows by $t' \to 0$. Since the opposite inequality $s(\Delta + u;\overline{m}) \geq s(\Delta;\overline{m})$ also holds similarly, we can show that $s(\Delta + u;\overline{m}) = s(\Delta;\overline{m})$.

For $t_1, t_2 \in \mathbb{Q}$ such that $0 < t_1 \leq t \leq t_2$, there exists inclusions $t_1(\Delta - u') \subset t(\Delta - u') \subset t_2(\Delta - u')$. Thus we have

$$s(t_1(\Delta - u');\overline{m}) \leq s(t(\Delta - u');\overline{m}) \leq s(t_2(\Delta - u');\overline{m}).$$

By Remark 4.3 (3) and the first equality of this lemma, which we have already proved, $s(t(\Delta - u');\overline{m}) = s(t\Delta;\overline{m})$ and $s(t_i(\Delta - u');\overline{m}) = t_i \cdot s(\Delta;\overline{m})$ for $i = 1, 2$. Substituting these inequalities, it holds that

$$t_1 \cdot s(\Delta;\overline{m}) \leq s(t\Delta;\overline{m}) \leq t_2 \cdot s(\Delta;\overline{m}).$$

By $t_1, t_2 \to t$, we have $s(t\Delta;\overline{m}) = t \cdot s(\Delta;\overline{m})$. □

Lemma 4.6. *Let $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_i \subset \cdots$ be an increasing sequence of convex sets in $\mathbb{R}^n$ and set $\Delta = \bigcup_{i=1}^\infty \Delta_i$. Then $s(\Delta;\overline{m}) = \sup_i s(\Delta_i;\overline{m}) = \lim_i s(\Delta_i;\overline{m})$ for any $\overline{m} \in \mathbb{R}_+^r$.*

Proof. This lemma follows from Lemma 3.7 immediately. □

Lemma 4.7. *Let Δ, Δ' be convex sets in $\mathbb{R}^n$. Then the following hold:*

$$s(\Delta;\overline{m}) + s(\Delta';\overline{m}) \leq s(\Delta + \Delta';\overline{m}), s(\Delta) \leq \sqrt[n]{n! \operatorname{vol}(\Delta)/|\overline{m}|_n}.$$

Proof. This lemma follows from $V_{k\Delta} \cdot V_{k\Delta'} \subset V_{k(\Delta+\Delta')}, \operatorname{vol}(W_{\Delta,\bullet}) = n! \operatorname{vol}(\Delta)$, and Lemma 3.10 (3), (4). □

5. Okounkov bodies and Seshadri constants

In this section, we show Theorem 1.1, which states that the Okounkov bodies give lower bounds of Seshadri constants.

5.1. **Definition of Okounkov bodies.**

For a subsemigroup Γ in $\mathbb{N} \times \mathbb{N}^n$ and $k \in \mathbb{N}$, set

$$\begin{aligned} \Delta(\Gamma) &= \Sigma(\Gamma) \cap (\{1\} \times \mathbb{R}^n), \\ \Gamma_k &= \Gamma \cap (\{k\} \times \mathbb{N}^n). \end{aligned}$$

Recall that $\Sigma(\Gamma)$ is the closed convex cone in $\mathbb{R} \times \mathbb{R}^n$ spanned by Γ. We consider $\Delta(\Gamma)$ and Γ_k as subsets in $\mathbb{R}^n$ and $\mathbb{N}^n$ respectively in a natural way.

Definition 5.1. We say a semigroup Γ in $\mathbb{N} \times \mathbb{N}^n$ is birational if

i) $\Gamma_0 = \{0\} \in \mathbb{N}^n$,

ii) Γ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group.

These conditions are (2.3) and (2,5) in [LM] respectively. Note that Γ is birational if and only if so is the graded linear series $W(\Gamma)_\bullet = \{W(\Gamma)_k\}_k := \{V_{\Gamma_k}\}_k$ on $(\mathbb{C}^\times)^n$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$.

Fix a monomial order $>$ on $\mathbb{N}^n$, i.e., $>$ is a total order on $\mathbb{N}^n$ such that (i) for every $u \in \mathbb{N}^n \setminus 0$, $u > 0$ holds, and (ii) if $v > u$ and $w \in \mathbb{N}^n$, then $w + v > w + u$. In this paper, $v > u$ does not contain the case $v = u$. Let X be a variety of dimension n and $z = (z_1, \ldots, z_n)$ a local coordinate system at a smooth point $p \in X$. (In [LM], they use "admissible flags" instead of local coordinate systems, but essentially there is no difference if $>$ is the lexicographic order. Local coordinate systems are used in [BC] for instance.) Then we obtain a valuation

$$\nu = \nu_{z,>} : \mathcal{O}_{X,p} \setminus \{0\} \to \mathbb{N}^n$$

as follows: for $f \in \mathcal{O}_{X,p} \setminus \{0\}$, we expand it as a formal power series

$$f = \sum_{u \in \mathbb{N}^n} c_u z^u$$

and set

$$\nu(f) = \min\{u \mid c_u \neq 0\},$$

where the minimum is taken with respect to the monomial order $>$.

Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on X. Then one can define the Okounkov body of $W_\bullet$ as follows:

Fix an isomorphism $L_p \cong \mathcal{O}_{X,p}$. Then this isomorphism naturally induces $L_p^{\otimes k} \cong \mathcal{O}_{X,p}$ for any $k \geq 0$ and we have the following map

$$W_k \setminus \{0\} \hookrightarrow L_p^{\otimes k} \setminus \{0\} \cong \mathcal{O}_{X,p} \setminus \{0\} \xrightarrow{\nu} \mathbb{N}^n.$$

We write $\nu(W_k) \subset \mathbb{N}^n$ for the image of $W_k \setminus \{0\}$ by this map. Note that $\nu(W_k)$ does not depend on the choice of the isomorphism $L_p \cong \mathcal{O}_{X,p}$ because ν maps any unit element in $\mathcal{O}_{X,p}$ to $0 \in \mathbb{N}^n$. Then

$$\Gamma_{W_\bullet} = \Gamma_{W_\bullet,z,>} := \bigcup_{k\in\mathbb{N}} \{k\} \times \nu(W_k) \subset \mathbb{N} \times \mathbb{N}^n$$

is a semigroup by construction. We define the Okounkov body of $W_\bullet$ with respect to z and $>$ by

$$\Delta(W_\bullet) = \Delta_{z,>}(W_\bullet) := \Delta(\Gamma_{W_\bullet}).$$

Thus $\Delta(W_\bullet)$ is an n-dimensional closed convex set in $\mathbb{R}^n$ because $\Gamma_{W_\bullet}$ is birational as we will prove in Lemma 5.2 later. In general, $\Delta(W_\bullet)$ is not bounded, in particular is not a convex body (= an n-dimensional compact convex set in $\mathbb{R}^n$), even if $W_\bullet$ is of finite dimensional type. But we call it Okounkov body according to custom.

In [LM, Lemma 2.12], they assume that $W_\bullet$ satisfies "condition (C)", which seems to be a little stronger condition than being birational(= Condition (B) in [LM]), to show that $\Gamma_{W_\bullet,z,>}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group for any z. But we can show that it is enough to assume that $W_\bullet$ is birational:

Lemma 5.2. *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a variety X. Then $\Gamma_{W_\bullet,z,>}$ is birational for any local coordinate system z at any smooth point $p \in X$ and any monomial order $>$.*

Proof. The condition i) in Definition 5.1 is clearly satisfied. So it is enough to show that $\Gamma_{W_\bullet}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group. Suppose that z is a local coordinate system and $>$ is a monomial order. Fix $k \gg 0$. Then the function field $K(X)$ is generated by $\{\, f/g \in K(X) \,|\, f, g \in W_k, g \neq 0\}$ over $\mathbb{C}$ because $W_\bullet$ is birational. Hence any $F \in K(X) \setminus \{0\}$ is written as $F = G/H$, where G, H are written as some polynomials over $\mathbb{C}$ of some elements in $\{\, f/g \in K(X) \,|\, f, g \in W_k, g \neq 0\}$. Therefore we can write as $F = G'/H'$ for some $G', H' \in W_{kl}$ for some $l \in \mathbb{Z}_+$. Thus $\nu(F) = \nu(G') - \nu(H') \in \nu(W_{kl}) - \nu(W_{kl}) \subset \mathbb{Z}^n$. Since the valuation $\nu : K(X) \setminus \{0\} \to \mathbb{Z}^n$ is surjective (note that ν is naturally extended to $K(X) \setminus \{0\}$), the group $\mathbb{Z}^n$ is generated by $\{\nu(W_{kl}) - \nu(W_{kl})\}_{l\in\mathbb{N}}$. Thus the subgroup $\{0\} \times \mathbb{Z}^n$ in $\mathbb{Z} \times \mathbb{Z}^n$ is generated by $\{0\} \times \{\nu(W_{kl}) - \nu(W_{kl})\}_l \subset \Gamma_{W_\bullet} - \Gamma_{W_\bullet}$.

On the other hand, $s_k \in W_k \setminus \{0\}$ and $s_{k+1} \in W_{k+1} \setminus \{0\}$ induce the element $(1, \nu(s_{k+1}) - \nu(s_k)) \in \Gamma_{W_\bullet} - \Gamma_{W_\bullet} \subset \mathbb{Z} \times \mathbb{Z}^n$. Since the group $\mathbb{Z} \times \mathbb{Z}^n$ is generated by $\{0\} \times \mathbb{Z}^n$ and $(1, \nu(s_{k+1}) - \nu(s_k))$, the semigroup $\Gamma_{W_\bullet}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group. $\square$

5.2. **Proof of Theorem 1.1.** At first, we state the idea of the proof. Let X be a projective variety and L a line bundle on X. Suppose that $V \subset H^0(X, L)$ is a subspace and set $W_\bullet = \{W_k\}$ by $W_k = V^k \subset H^0(X, kL)$. Anderson [An] proved that $(\operatorname{Proj} \bigoplus_k V^k, \mathcal{O}(1))$ degenerates to the toric variety $(\operatorname{Proj} \mathbb{C}[\Gamma_{W_\bullet}], \mathcal{O}(1))$ if $\Gamma_{W_\bullet}$ is finitely generated as a semigroup. From this result and the lower semicontinuities of Seshadri constants, Theorem 1.1 for $W_\bullet = \{V^k\}_k$ is easily shown if $\Gamma_{W_\bullet}$ is finitely generated. But it seems that $\Gamma_{W_\bullet}$ is seldom finitely generated, even if $W_\bullet = \{H^0(X, kL)\}_k$ for a very ample L on X (cf.[LM, Lemma 1.7]).

Thus we modify the above idea. Instead of the degeneration of the variety or the section ring $\bigoplus_k W_k$, we degenerate W_k for each k severally. This is one of the main reason why we define Seshadri constants by using jet separations, instead of the usual definition by blowing ups as Lemmas 3.11 and 3.12.

Lemma 5.3. *Let $\Gamma \subset \mathbb{N} \times \mathbb{N}^n$ be a birational semigroup. Then*

$$s(\Delta(\Gamma); \overline{m}) = \sup \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}$$

holds for any $\overline{m} \in \mathbb{R}^r_+$. In other words, $\varepsilon(W_{\Delta(\Gamma),\bullet}; \overline{m}) = \varepsilon(W(\Gamma)_\bullet; \overline{m})$ holds.

Proof. By definition, it holds that

$$\varepsilon(W_{\Delta(\Gamma),\bullet}; \overline{m}) = s(\Delta(\Gamma); \overline{m}) = \sup \frac{\tilde{s}(k\Delta(\Gamma); \overline{m})}{k} = \lim \frac{\tilde{s}(k\Delta(\Gamma); \overline{m})}{k}$$

and

$$\varepsilon(W(\Gamma)_\bullet; \overline{m}) = \sup \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}.$$

Furthermore, Γ_k is contained in $k\Delta(\Gamma)$ for any k. Thus $\varepsilon(W_{\Delta(\Gamma),\bullet}; \overline{m}) \geq \varepsilon(W(\Gamma)_\bullet; \overline{m})$ is clear. Hence it is enough to show

$$\lim_k \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} \geq \lim \frac{\tilde{s}(k\Delta(\Gamma); \overline{m})}{k}.$$

When Γ is finitely generated, there exists $\tilde{u} = (l, u) \in \Gamma \subset \mathbb{N} \times \mathbb{N}^n$ such that $(\Sigma(\Gamma) + \tilde{u}) \cap (\mathbb{N} \times \mathbb{N}^n) \subset \Gamma$ by [Kh, §3, Proposition 3] (see also [LM, Subsection 2.1]). This gives the inclusion $(k\Delta(\Gamma) \cap \mathbb{N}^n) + u \subset \Gamma_{k+l}$, so $\tilde{s}(k\Delta(\Gamma); \overline{m}) = \tilde{s}((k\Delta(\Gamma) \cap \mathbb{N}^n) + u; \overline{m}) \leq \tilde{s}(\Gamma_{k+l}; \overline{m})$ holds. Thus we have

$$\begin{aligned} \lim_k \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} &= \lim_k \frac{\tilde{s}(\Gamma_{k+l}; \overline{m})}{k} \\ &\geq \lim_k \frac{\tilde{s}(k\Delta(\Gamma); \overline{m})}{k}. \end{aligned}$$

In general case, we take an increasing sequence $\Gamma^1 \subset \Gamma^2 \subset \cdots \subset \Gamma$ such that each Γ^i is a finitely generated birational subsemigroup of Γ and $\bigcup_{i\geq 1} \Gamma_i = \Gamma$. Then it is easy to show $\bigcup_i \Delta(\Gamma^i)^\circ = \Delta(\Gamma)^\circ$. By Lemmas 4.4 and 4.6, we have

$$s(\Delta(\Gamma); \overline{m}) = s(\Delta(\Gamma)^\circ; \overline{m}) = \lim s(\Delta(\Gamma^i)^\circ; \overline{m}) = \lim s(\Delta(\Gamma^i); \overline{m}).$$

Since each Γ^i is finitely generated, we can apply this lemma, Lemma 5.3 to Γ^i. So we have

$$s(\Delta(\Gamma^i); \overline{m}) = \lim_k \frac{\tilde{s}(\Gamma^i_k; \overline{m})}{k} \leq \lim_k \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}.$$

Thus $s(\Delta(\Gamma); \overline{m}) = \lim s(\Delta(\Gamma^i); \overline{m}) \leq \lim_k \dfrac{\tilde{s}(\Gamma_k; \overline{m})}{k}$ holds. □

Broadly speaking, the geometrical meaning of Lemma 5.3 is that Seshadri constants of ample line bundles (on non-normal toric varieties) at very general points does not change by normalizations. In fact, when Γ is finitely generated, $(\mathrm{Proj} \bigoplus_k W_{\Delta(\Gamma),k}, \mathcal{O}(1)) = (\mathrm{Proj}\, \mathbb{C}[\Sigma(\Gamma) \cap (\mathbb{N} \times \mathbb{Z}^n)], \mathcal{O}(1))$ is nothing but the normalization of the toric variety $(\mathrm{Proj} \bigoplus_k W(\Gamma)_k, \mathcal{O}(1)) = (\mathrm{Proj}\, \mathbb{C}[\Gamma], \mathcal{O}(1))$.

To prove Proposition 5.6, which is the key of the proof of Theorem 1.1, we need one more lemma about monomial orders.

Lemma 5.4. *Let $>$ be a monomial order on $\mathbb{N}^n$ and S a finite set in $\mathbb{N}^n$. Then there exists a vector $\alpha \in \mathbb{Z}^n_+$ satisfying the following:*

If $v > u$ for $u \in S$ and $v \in \mathbb{N}^n$, then $\alpha \cdot v > \alpha \cdot u$ holds, where $\alpha \cdot u, \alpha \cdot v$ are the usual inner products.

Proof. For each $u \in S$, set $S_u = \{v \in \mathbb{N}^n \mid v > u\}$. Let I_u be the ideal in the polynomial ring $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$ generated by $\{\, x^v \mid v \in S_u\}$. By Hilbert's basis theorem, I_u is generated by $x^{v_{u1}}, \ldots, x^{v_{uk_u}}$ for some $k_u \in \mathbb{N}$ and $v_{u1}, \ldots, v_{uk_u} \in S_u$. Therefore any $v \in S_u$ is contained in $v_{uj} + \mathbb{N}^n$ for some j.

We use the following fact (cf.[Ro, Theorem 2.5]):

Fact 5.5. *Let $>$ be a monomial order on $\mathbb{N}^n$. Then there exist an integer $s \geq 1$ with $1 \leq s \leq n$ and s vectors $u_1, ..., u_s \in \mathbb{R}^n$ which satisfy the following:*

For $u, v \in \mathbb{N}^n$, $v > u$ if and only if $\pi(v) >_{lex} \pi(u)$, where

$$\pi : \mathbb{N}^n \to \mathbb{R}^s; u \mapsto (u_1 \cdot u, \ldots, u_s \cdot u)$$

and $>_{lex}$ is the lexicographic order on $\mathbb{R}^s$.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Z}^n$ and consider the above $u_1, \dots, u_s$ and π. By the definition of the lexicographic order, for any $\gamma >_{lex} \delta$ in $\mathbb{R}^s$, $\beta \cdot \gamma > \beta \cdot \delta$ holds for $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{R}^s_+$ if $\beta_1 \gg \dots \gg \beta_n$ (of course, such β depends on γ and δ). As $\gamma >_{lex} \delta$, we consider $\pi(e_i) >_{lex} \pi(0) = 0$ and $\pi(v_{uj}) >_{lex} \pi(u)$ for $1 \leq i \leq n$, $u \in S$ and $1 \leq j \leq k_u$ and choose β as above. Since S is a finite set, we can take a common $\beta \in \mathbb{R}^s_+$. Since $\beta \cdot \pi(u) = (\beta_1 u_1 + \dots + \beta_s u_s) \cdot u$,

$$\alpha' \cdot e_i > 0, \ \alpha' \cdot v_{uj} > \alpha' \cdot u \cdots (*)$$

holds for $1 \leq i \leq n, u \in S$ and $1 \leq j \leq k_u$ if we take $\alpha' \in \mathbb{Q}^n$ sufficiently near $(\beta_1 u_1 + \cdots \beta_s u_s)$. Multiplying α' by a sufficiently divisible positive integer N, and set $\alpha := N\alpha' \in \mathbb{Z}^n$. By $(*)$, it follows that $\alpha \in \mathbb{Z}^n_+$ and $\alpha \cdot v_{uj} > \alpha \cdot u$ for $u \in S$ and j.

We show this α satisfies the condition in the statement of this lemma. Fix $u \in S$ and $v \in \mathbb{N}^n$ such that $v > u$, i.e., v is contained in S_u. Then $v \in v_{uj} + \mathbb{N}^n$ for some $1 \leq j \leq k_u$. Thus $\alpha \cdot v = \alpha \cdot v_{uj} + \alpha \cdot (v - v_{uj}) \geq \alpha \cdot v_{uj} > \alpha \cdot u$. $\square$

Now we show the key proposition. Roughly speaking, this state that $W \subset H^0(X, L)$ generically separates jets no less than $V_{\nu(W)}$:

Proposition 5.6. *Let L be a line bundle on an n-dimensional variety X and set $\nu = \nu_{z,>}$ be the valuation map defined by a local coordinate system $z = (z_1, \dots, z_n)$ at a smooth point $p \in X$ and a monomial order $>$ on $\mathbb{N}^n$.*

Then $j(W; \overline{m}) \geq \tilde{s}(\nu(W); \overline{m})$ holds for any subspace W of $H^0(X, L)$ and any $\overline{m} \in \mathbb{R}^r_+$.

Proof. By considering an increasing sequence of finite dimensional subspaces in W, we may assume that W is finite dimensional.

Let $\pi : U \to \mathbb{C}^n$ be the étale morphism defined by $z_1, \dots, z_n$ in an open neighborhood $U \subset X$ of p. By the morphism π, we can identify $\mathcal{O}^{an}_{X,p}$ with $\mathcal{O}^{an}_{\mathbb{C}^n,0} = \mathbb{C}\{x_1, \dots, x_n\}$, where $x_1, \dots, x_n$ are the coordinates on $\mathbb{C}^n$ such that $\pi^* x_i = z_i$. Then we can regard W as a subspace in $\mathbb{C}\{x_1, \dots, x_n\}$ by $W \hookrightarrow L_p \cong \mathcal{O}_{X,p} \hookrightarrow \mathbb{C}\{x_1, \dots, x_n\}$. Note that ν is extended to $\mathcal{O}^{an}_{X,p} \setminus \{0\} = \mathbb{C}\{x_1, \dots, x_n\} \setminus \{0\} \to \mathbb{N}^n$ naturally.

Choose and fix $f_u \in \nu^{-1}(u) \cap W$ for each $u \in \nu(W)$. Then it holds that $V = \bigoplus_{u \in \nu(W)} \mathbb{C} f_u$ because $\# \nu(W) = \dim W$ (cf.[LM] or [BC]). Since $\nu(W)$ is a finite set, there exists $\alpha \in \mathbb{Z}^n_+$ satisfying the following by Lemma 5.4: if $v > u$ for $u \in \nu(W)$ and $v \in \mathbb{N}^n$, it holds that $\alpha \cdot v > \alpha \cdot u$.

The vector α induces the action $\circ$ of $\mathbb{C}^\times$ on $\mathbb{C}\{x_1, \dots, x_n\}$ by $t \circ x^u = t^{\alpha \cdot u} x^u$ for $t \in \mathbb{C}^\times$ and $u \in \mathbb{N}^n$. For $f_u = \sum_v c_{uv} z^v = \sum_v c_{uv} x^v$ (note we

identify z_i and x_i), the regular function

$$\frac{t \circ f_u}{t^{\alpha\cdot u}} = t^{-\alpha\cdot u}\sum_v c_{uv}t^{\alpha\cdot v}x^v = \sum_v c_{uv}t^{\alpha\cdot v-\alpha\cdot u}x^v$$

on a neighborhood of $\mathbb{C}^\times \times \{0\}$ is naturally extended to a regular function on a neighborhood $\mathcal{U}$ (in $\mathbb{C}\times\mathbb{C}^n$) of $\mathbb{C}\times\{0\}$. Note that $\alpha\cdot v - \alpha\cdot u \geq 0$ if $c_{uv}\neq 0$. We denote the regular function by F_u. Set $\mathcal{W} = \bigoplus_{u\in\nu(W)}\mathbb{C}F_u$.

We prove this proposition only for $r=1$. When $r>1$, the proof is similar. So we leave the details to the reader.

By the assumption that $r=1$, we may assume $\overline{m}=1$. Choose a very general section σ of the first projection $\mathcal{U}\to\mathbb{C}$. Note that $\sigma(0)$ is contained in $\{0\}\times(\mathbb{C}^\times)^n \cong (\mathbb{C}^\times)^n$. (When $r>1$, we choose very general sections $\sigma_1,\dots,\sigma_r$ of $\mathcal{U}\to\mathbb{C}$.) Let $\mathcal{I}$ be the ideal sheaf corresponding to $\sigma(\mathbb{C})$ on $\mathcal{U}\subset\mathbb{C}\times\mathbb{C}^n$, and consider the following map between flat sheaves over $\mathbb{C}$:

$$\phi : \mathcal{W}\otimes_{\mathbb{C}}\mathbb{C}\{t\} \to \mathcal{O}_{\mathcal{U}}^{an} \to \mathcal{O}_{\mathcal{U}}^{an}/\mathcal{I}^{j+1}$$

for $j\geq 0$. We denote $W_t := \mathcal{W}\otimes\mathbb{C}\{t\}|_{\{t\}\times\mathbb{C}^n}$ and

$$\phi_t := \phi|_{\{t\}\times\mathbb{C}^n} : W_t \to \mathcal{O}_{\mathcal{U}}^{an}/\mathcal{I}^{j+1}|_{\{t\}\times\mathbb{C}^n} = \mathcal{O}_{\mathbb{C}^n}/\mathfrak{m}_{\sigma(t)}^{j+1}$$

for $t\in\mathbb{C}$. By the flatness, ϕ_t is surjective for very general t if so is ϕ_0. Thus if W_0 separates m-jets at $\sigma(0)$ for $m\in\mathbb{Z}$, then W_t also separates m-jets at $\sigma(t)$ for very general t. Since we choose a very general section, it follows that

$$j(W_t)\geq j(W_0)$$

for t in a neighborhood of 0. Note $j(W_t)$ can be defined similarly in this analytic setting. Since there is a natural identification of W_t and W for $t\in\mathbb{C}^\times$ by the action $\circ$, we have $j(W)=j(W_t)$. Note that $j(W)$ does not depend on whether we consider W as a subspace in $H^0(X,L)$ or in $\mathbb{C}\{x_1,\dots,x_n\}$ because $\pi : U\to\mathbb{C}^n$ is étale. On the other hand, $W_0 = V_{\nu(W)}\subset\mathbb{C}[x_1,\dots,x_n]$ since $F_u = c_{uu}x^u + t\cdot$ higher term, $c_{uu}\neq 0$. Thus $j(W_0)=\tilde{s}(\nu(W))$ holds by the definition of $\tilde{s}(\cdot)$. From these inequalities, we have $j(W)\geq\tilde{s}(\nu(W))$. □

Now we can show the main theorem easily:

Theorem 5.7 (=Theorem 1.1). *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on an n-dimensional variety X. Fix a local coordinate system $z=(z_1,\dots,z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$.*

Then $\varepsilon(W_\bullet;\overline{m})\geq s(\Delta_{z,>}(W_\bullet);\overline{m})$ holds for any $r\in\mathbb{Z}_+$ and $\overline{m}\in\mathbb{R}^r_+$.

Proof. Let $\Gamma := \Gamma_{W_\bullet,z,>} \subset \mathbb{N} \times \mathbb{N}^n$ be the semigroup defined by $W_\bullet, z$ and $>$ in the definition of Okounkov bodies. Then $\Gamma_k = \nu(W_k) \subset \mathbb{N}^n$ and $\Delta(\Gamma) = \Delta_{z,>}(W_\bullet)$ by definition. By Proposition 5.6, we have

$$\varepsilon(W_\bullet; \overline{m}) = \lim \frac{j(W_k; \overline{m})}{k} \geq \lim \frac{\tilde{s}(\nu(W_k); \overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}.$$

Since Γ is birational by Lemma 5.2, it holds that

$$\lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} = s(\Delta(\Gamma); \overline{m}) = s(\Delta_{z,>}(W_\bullet); \overline{m})$$

from Lemma 5.3. Thus $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m})$ holds. □

Remark 5.8. Since $s(\Delta(W_\bullet); \overline{m}) = \varepsilon(W_{\Delta(W_\bullet),\bullet}; \overline{m})$, the above theorem says that the Seshadri constant of $W_\bullet$ is greater than or equal to that of $W_{\Delta(W_\bullet),\bullet}$. For a convex set $\Delta \subset \mathbb{R}^r_{\geq 0}$, $W_{\Delta,\bullet}$ is considered as a graded linear series on $\mathbb{C}^n$. Note that $\Delta_{z,>}(W_{\Delta,\bullet})$ is nothing but Δ itself for the standard local coordinate system $z = (x_1, \ldots, x_n)$ at $0 \in \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ (cf.[LM, Proposition 6.1]). Hence Seshadri constants are minimal in monomial (or toric) cases for a fixed Okounkov body.

See [LM, Remark 5.5] for another relation between Okounkov bodies and Seshadri constants, though the relation is not written explicitly there. Note that the relation also holds for a birational graded linear series.

6. Computations and estimations of $s(\Delta; \overline{m}), \tilde{s}(S; \overline{m})$

6.1. **In the case $r = 1$.** It is very hard to compute $\tilde{s}(S; \overline{m})$ in general, but when $r = 1$, we can compute it by considering the rank of matrices. In this subsection, we mainly consider S or Δ which is bounded and contained in $\mathbb{R}^n_{\geq 0}$. For general S and Δ, we can reduce them to such cases by Remark 4.3 and Lemma 4.6.

For $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, we define a natural number $\begin{bmatrix} u \\ \lambda \end{bmatrix} \in \mathbb{N}$ to be

$$\begin{bmatrix} u \\ \lambda \end{bmatrix} = \prod_{k=1}^{n} \binom{u_k}{\lambda_k},$$

where $\displaystyle\binom{u_k}{\lambda_k} = \frac{u_k(u_k - 1) \cdots (u_k - \lambda_k + 1)}{\lambda_k!}$ is the binomial coefficient.

Set $1_n = (1, \ldots, 1)$ in $(\mathbb{C}^\times)^n$ and let $\mathfrak{n}$ be an $\mathfrak{m}_{1_n}$-primary ideal on $(\mathbb{C}^\times)^n$. Assume that $\mathfrak{n}$ is generated by monomials of $x_1 - 1, \ldots, x_n - 1$. In particular, there exists a finite subset $\Phi_{\mathfrak{n}} \subset \mathbb{N}^n$ such that $\mathfrak{n}$ is the

restriction of

$$\bigoplus_{\lambda\in\mathbb{N}^n\setminus\Phi_{\mathfrak{n}}} \mathbb{C}(x-1_n)^\lambda \subset \mathcal{O}_{\mathbb{C}^n}$$

on $(\mathbb{C}^\times)^n$, where $(x-1_n)^\lambda = (x_1-1)^{\lambda_1}\cdots(x_n-1)^{\lambda_n}$.

For a bounded subset S in $\mathbb{R}^n_{\geq 0}$, we set a matrix $A_{S,\mathfrak{n}}$ by

$$A_{S,\mathfrak{n}} = \left(\begin{bmatrix} u \\ \lambda \end{bmatrix}\right)_{(\lambda,u)\in\Phi_{\mathfrak{n}}\times(S\cap\mathbb{N}^n)}.$$

When $\mathfrak{n} = \mathfrak{m}_{1_n}^{m+1}$ for $m\in\mathbb{N}$, we denote $A_{S,\mathfrak{m}_{1_n}^{m+1}}$ by $A_{S,m}$ for short.

The following Proposition is a straightforward generalization of results in [Du, Proposition 13] and [BBC+, 3.10], and the proof is same. We prove it here for the convenience of the reader:

Proposition 6.1. *Let S be a bounded set in $\mathbb{R}^n_{\geq 0}$ and $\mathfrak{n}$ an $\mathfrak{m}_{1_n}$-primary ideal generated by monomials of $x_1-1,\dots,x_n-1$. Then the following conditions are equivalent:*

i) *the natural map $\varphi_{S,\mathfrak{n}} : V_S \to \mathcal{O}_{(\mathbb{C}^\times)^r}/\mathfrak{n} = \bigoplus_{\lambda\in\Phi_{\mathfrak{n}}}\mathbb{C}(x-1_n)^\lambda\,;\, f\mapsto f+\mathfrak{n}$ is not surjective,*

ii) $\operatorname{rank} A_{S,\mathfrak{n}} < \#\Phi_{\mathfrak{n}}$.

Furthermore, if $\#(S\cap\mathbb{N}^n) = \#\Phi_{\mathfrak{n}}$ then these are equivalent to

iii) *there exists a nonzero element $f\in\bigoplus_{\lambda\in\Phi_{\mathfrak{n}}}\mathbb{R}u^\lambda\subset\mathbb{R}[u_1,\dots,u_n]$ such that $S\cap\mathbb{N}^n$ is contained in the hypersurface in $\mathbb{R}^n$ defined by f.*

Proof. Since $A_{S,\mathfrak{n}}$ is the matrix of $\varphi_{S,\mathfrak{n}}$ with respect to bases $\{x^u\}_{u\in S\cap\mathbb{N}^n}$ and $\{(x-1_n)^\lambda\}_{\lambda\in\Phi_{\mathfrak{n}}}$, the equivalence of i) and ii) is clear.

We assume $\#(S\cap\mathbb{N}^n) = \#\Phi_{\mathfrak{n}}$ and show the equivalence of ii) and iii). By definition, it holds that

$$\begin{bmatrix} u \\ \lambda \end{bmatrix} = \frac{1}{\lambda_1!\cdots\lambda_n!}\prod_{k=1}^{n}\prod_{l=0}^{\lambda_k-1}(u_k-l).$$

Hence $\operatorname{rank} A_{S,\mathfrak{n}} = \operatorname{rank}(\prod_{k=1}^n\prod_{l=0}^{\lambda_k-1}(u_k-l))_{(\lambda,u)}$. For $\lambda\in\Phi_{\mathfrak{n}}$, any $\lambda'\in\mathbb{N}^n$ satisfying $\lambda-\lambda'\in\mathbb{N}^n$ is also contained in $\Phi_{\mathfrak{n}}$ because $\mathfrak{n}$ is an ideal. From this, the matrix $(\prod_{k=1}^n\prod_{l=0}^{\lambda_k-1}(u_k-l))_{(\lambda,u)}$ changes to the matrix $(u^\lambda)_{(\lambda,u)} = ({u_1}^{\lambda_1}\cdots{u_n}^{\lambda_n})_{(\lambda,u)}$ by a suitable sequence of row operations.

Thus $A_{S,\mathfrak{n}}$ is not regular if and only if rows of $(u^\lambda)_{(\lambda,u)}$ are linearly dependent, and this is equivalent to iii). □

Remark 6.2. If $\mathfrak{n} = \mathfrak{m}_{1_n}^{m+1}$ for $m\in\mathbb{N}$, then $\Phi_{\mathfrak{m}_{1_n}^{m+1}} = \{\lambda = (\lambda_1,\dots,\lambda_n)\in\mathbb{N}^n \mid \lambda_1+\ldots+\lambda_n\leq m\}$. So in this case, iii) means that $S\cap\mathbb{N}^n$ is contained in a hypersurface of degree m in $\mathbb{R}^n$.

Corollary 6.3. *For a bounded set S in $\mathbb{R}^n_{\geq 0}$ and a bounded convex set Δ in $\mathbb{R}^n_{\geq 0}$, it holds that*

$$\begin{aligned}\tilde{s}(S) &= \max\left\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{S,m} = \binom{m+n}{n}\right\},\\ s(\Delta) &= \sup_{k\in\mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k}\\ &= \lim_{k\in\mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k}.\end{aligned}$$

Proof. This corollary follows from Lemma 3.8, Proposition 6.1 and $\#\Phi_{\mathfrak{m}_{1_n}^{m+1}} = \binom{m+n}{n}$. □

Remark 6.4. By Corollary 6.3, we can compute $\tilde{s}(S)$ by finite calculations. In fact $A_{S,m}$ is a $\binom{m+n}{n} \times \#(S\cap\mathbb{N}^n)$ matrix, so $\operatorname{rank} A_{S,m} = \binom{m+n}{n}$ only if $\binom{m+n}{n} \leq \#(S\cap\mathbb{N}^n)$. Note that $\#(S\cap\mathbb{N}^n)$ is finite by the boundedness of S.

By Corollary 6.3, we can describe the Seshadri constant of a polarized toric variety at a very general point as the supremum or the limit of computable numbers as follows:

Let P be an integral polytope in $\mathbb{R}^n$ of dimension n. Then we can define a polarized toric variety (X_P, L_P) as

$$(X_P, L_P) = (\operatorname{Proj} \bigoplus_{k\in\mathbb{N}} V_{kP}, \mathcal{O}(1)).$$

In other words,

$$(X_P, L_P) = (\operatorname{Proj} \mathbb{C}[\Gamma_P], \mathcal{O}(1)),$$

where $\Gamma_P = \Sigma(\{1\} \times P) \cap (\mathbb{N} \times \mathbb{Z}^n)$ is a subsemigroup of $\mathbb{N} \times \mathbb{Z}^n$. Note that $\bigoplus_{k\in\mathbb{N}} V_{kP} = \mathbb{C}[\Gamma_P]$ is naturally graded by $\mathbb{N}$. The following corollary is essentially stated in [BBC+, 3.10], at least in case of $n = 2$:

Corollary 6.5. *Let P be an integral polytope of dimension n contained in $\mathbb{R}^n_{\geq 0}$. Then it holds that*

$$\begin{aligned}j(L_P) &= \max\left\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{P,m} = \binom{m+n}{n}\right\},\\ \varepsilon(X_P, L_P; 1) &= \sup_{k\in\mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k}\\ &= \lim_{k\in\mathbb{Z}_+} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k}.\end{aligned}$$

Proof. Note that there exists a natural embedding $(\mathbb{C}^\times)^n \hookrightarrow X_P$ and an identification of $W_{P,k} = V_{kP}$ and $H^0(X_P, kL_P)$. Thus it clearly holds that $j(L_P) = j(V_P) = \tilde{s}(P)$ and $\varepsilon(X_P, L_P; 1) = \varepsilon((\mathbb{C}^\times)^n, W_{P,\bullet}; 1) = s(P)$ (or more generally $j(L_P; \overline{m}) = \tilde{s}(P; \overline{m})$ and $\varepsilon(X_P, L_P; \overline{m}) = \varepsilon((\mathbb{C}^\times)^n, W_{P,\bullet}; \overline{m}) = s(P; \overline{m})$ for any $\overline{m} \in \mathbb{R}^r_+$). So this corollary follows from Corollary 6.3. □

6.2. **In the case $r > 1$.** In the above subsection, we investigate $s(\Delta)$ or $\tilde{s}(S)$. Now we consider $s(\Delta; \overline{m})$ for general $\overline{m}$. At least there are three methods to estimate $s(\Delta; \overline{m})$ or $\tilde{s}(S; \overline{m})$ from below (though they are not enough to obtain good estimations in general). These methods may be known to specialists, at least for $S \subset \mathbb{Z}^2$ or $\Delta \subset \mathbb{R}^2$. But we state here in our settings for the convenience of the reader.

The first one uses degenerations of ideals:

Proposition 6.6. *Let $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Assume that there exists a flat family $\{\mathfrak{n}_t\}_{t \in T}$ of ideals on $(\mathbb{C}^\times)^n$ over a smooth curve T such that*

i) *$\mathfrak{n}_t = \mathfrak{m}_{p_{1,t}}^{m_1+1} \otimes \cdots \otimes \mathfrak{m}_{p_{r,t}}^{m_r+1}$ for general $t \in T$, where $p_{1,t}, \ldots, p_{r,t}$ are distinct r points in $(\mathbb{C}^\times)^n$,*
ii) *$\mathfrak{n}_0$ is an $\mathfrak{m}_{1_n}$-primary ideal generated by monomials of $x - 1_n$ for $0 \in T$.*

Then V_S generically separates $\overline{m}$-jets for a bounded subset $S \subset \mathbb{R}^n_{\geq 0}$ if $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$.

Proof. Let $\mathcal{I}$ be the ideal on $(\mathbb{C}^\times)^n \times T$ corresponding to the family $\{\mathfrak{n}_t\}_{t \in T}$ and assume $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$.

Consider the natural map

$$\phi : V_S \otimes_{\mathbb{C}} \mathcal{O}_T \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}.$$

By the assumption $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$ and Proposition 6.1,

$$\phi_0 : V_S = V_S \otimes \mathcal{O}_T|_{(\mathbb{C}^\times)^n \times \{0\}} \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}|_{(\mathbb{C}^\times)^n \times \{0\}} = \mathcal{O}_{(\mathbb{C}^\times)^n}/\mathfrak{n}_0$$

is surjective. Thus

$$\phi_t : V_S \otimes \mathcal{O}_T|_{(\mathbb{C}^\times)^n \times \{t\}} \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}|_{(\mathbb{C}^\times)^n \times \{t\}} = \mathcal{O}_{(\mathbb{C}^\times)^n}/\mathfrak{n}_t$$

is also surjective for general $t \in T$ by the flatness. This means V_S separates $\overline{m}$-jets at $p_{1,t}, \ldots, p_{r,t}$ by i). In particular V_S generically separates $\overline{m}$-jets. □

Example 6.7. Set $T = \mathbb{C}$, $\overline{m} = (3, 1) \in \mathbb{N}^2$, and $\mathfrak{n}_t = (x-1, y-1)^4(x-1-t, y-1)^2$ for $t \in \mathbb{C}^\times = T \setminus 0$. Then the family of ideals $\{\mathfrak{n}_t\}_{t \in \mathbb{C}^\times}$ extends over $\mathbb{C}$ by $\mathfrak{n}_0 = ((x-1)^6, (x-1)^4(y-1), (x-1)^2(y-1)^2, (x-1)(y-1)^3, (y-1)^4)$. Since the family $\{\mathfrak{n}_t\}_{t \in \mathbb{C}}$ is flat near 0, we

can apply Proposition 6.6. In Figure 1 bellow, five $\circ$ correspond to the monomial generators of $\mathfrak{n}_0$, and thirteen $\bullet$ are all points in $\Phi_{\mathfrak{n}_0}$. Set S be thirteen $\bullet$ in Figure 2. Then rank $A_{S,\mathfrak{n}_0} = 13 = \dim \mathcal{O}/\mathfrak{n}_0$. Therefore V_S generically separates $(3,1)$-jets, i.e., $\tilde{s}(S;(3,1)) \geq 1$ holds.

Note that this can be shown by the method in [Du], as well.

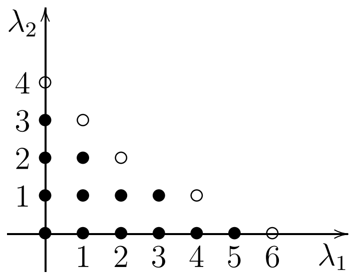


Figure 1

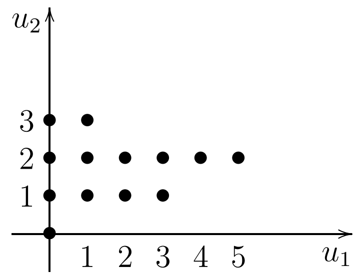


Figure 2

The second one uses finite morphisms induced by changes of lattices (cf.[Ga, Lemma 2.1]):

Proposition 6.8. *Let $\iota : \mathbb{Z}^n \to \mathbb{Z}^n$ be an injection between abelian groups of same rank, whose degree is $d \in \mathbb{Z}_+$. For a convex set $\Delta \subset \mathbb{R}^n$, set $\Delta' = \iota_{\mathbb{R}}^{-1}(\Delta)$, where $\iota_{\mathbb{R}} = \iota \otimes id_{\mathbb{R}} : \mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R} \to \mathbb{Z}^n \otimes \mathbb{R} = \mathbb{R}^n$. Then,*

$$s(\Delta; \underbrace{\overline{m}, \ldots, \overline{m}}_{d}) \geq s(\Delta'; \overline{m})$$

holds for any $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}_+^r$.

Proof. By Lemmas 4.4 and 4.6, we may assume that Δ is a rational polytope. Furthermore we may assume Δ, Δ' are integral polytopes by Lemma 4.5.

The injective morphism ι induces the quotient morphism

$$\pi : X_\Delta \to X_{\Delta'}$$

such that $\pi^* L_{\Delta'} = L_\Delta$. Note that π is a finite morphism of degree d.

We choose very general points $p_1, \ldots, p_r$ in $X_{\Delta'}$, then $\bigcup_i \pi^{-1}(p_i)$ are smooth dr-points in X_Δ. Consider the following diagram:

$$\begin{tikzcd} \widetilde{X}_\Delta \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{X}_{\Delta'} \arrow[d, "\mu'"] \\ X_\Delta \arrow[r, "\pi"] & X_{\Delta'}, \end{tikzcd}$$

where μ is the blowing up along $\bigcup_i \pi^{-1}(p_i)$ and μ' is the blowing up along $\{p_1, \ldots, p_r\}$. Let E_{ij} be the exceptional divisor over p_{ij}, where $\pi^{-1}(p_i) = \{p_{i1}, \ldots, p_{id}\}$, and E'_i the exceptional divisor over p_i.

By Lemma 3.11, $\mu'^* L_{\Delta'} - s(\Delta'; \overline{m}) \sum_i m_i E'_i$ is nef. Thus $\mu^* L_\Delta - s(\Delta'; \overline{m}) \sum_{i,j} m_i E_{ij} = \widetilde{\pi}^*(\mu'^* L_{\Delta'} - s(\Delta'; \overline{m}) \sum_i m_i E'_i)$ is also nef. Since ampleness is an open condition in a flat family, $\mu^* L_\Delta - s(\Delta'; \overline{m}) \sum_{i,j} m_i E_{ij}$ is also nef if μ is the blowing up of X_Δ along very general dr-points. Thus $s(\Delta; \underbrace{\overline{m}, \ldots, \overline{m}}_{d}) \geq s(\Delta'; \overline{m})$ holds by Lemma 3.11. □

Example 6.9. Let $\Delta \subset \mathbb{R}^3$ be the convex hull of $(0,0,0), (1,1,0), (1,0,1)$, and $(0,1,1)$. Set $\iota : \mathbb{Z}^3 \to \mathbb{Z}^3$ by $\iota(e_1) = (1,1,0), \iota(e_2) = (1,0,1)$, and $\iota(e_3) = (0,1,1)$ for the standard basis e_1, e_2, e_3 of $\mathbb{Z}^3$. Then the degree of ι is 2, and $\Delta' = \iota_{\mathbb{R}}^{-1}(\Delta)$ is an integral polytope corresponding to $(\mathbb{P}^3, \mathcal{O}(1))$. Thus $s(\Delta; 1, 1) \geq s(\Delta'; 1) = 1$ holds by Proposition 6.8. In this case, $s(\Delta; 1, 1) = 1$ holds by Lemma 4.7.

The last one uses degenerations of varieties.

Let P be a polytope of dimension n in $\mathbb{R}^n$. A polytope decomposition $\mathcal{P}$ of P is a finite subset of $\{\sigma \,|\, \sigma \text{ is a polytope in } \mathbb{R}^n\}$ such that

i) $P = \bigcup_{\sigma \in \mathcal{P}} \sigma$,
ii) if $\sigma \in \mathcal{P}$ and τ is a face of σ, then $\tau \in \mathcal{P}$,
iii) if $\sigma, \sigma' \in \mathcal{P}$, then $\sigma \cap \sigma'$ is either a common face of σ, σ' or empty.

A decomposition $\mathcal{P}$ is integral if all $\sigma \in \mathcal{P}$ are integral polytopes.

For an integral polytope decomposition $\mathcal{P}$ of P, we say a function $\varphi : P \to \mathbb{R}$ is a lifting function of $\mathcal{P}$ if

i) φ is piecewise affine and strictly convex with respect to $\mathcal{P}$,
ii) φ takes an integral value at every $u \in P \cap M$.

If there exists a lifting function of an integral polytope decomposition $\mathcal{P}$, one can construct a toric degeneration of X_P, i.e., one can construct an $n+1$ dimensional toric variety $\mathcal{X}$, an ample line bundle $\mathcal{L}$ on $\mathcal{X}$, and a flat projective toric morphism $f : \mathcal{X} \to \mathbb{A}^1$ such that

- the central fiber is $X_0 = \bigcup_{i=1}^{r'} X_{P_i}$ and $\mathcal{L}|_{X_{P_i}} = L_{P_i}$,
- the general fiber is $X_t = X_P$ and $L_t = L_P$ for any $t \in \mathbb{A}^1 \setminus \{0\}$,

where $X_t = f^{-1}(t), L_t = \mathcal{L}|_{X_t}$ for $t \in \mathbb{A}^1$ and $\mathcal{P}^{[n]} := \{\sigma \in \mathcal{P} \,|\, \dim \sigma = n\} = \{P_1, \ldots, P_{r'}\}$. See [GS] for example. From this degeneration, we obtain lower bounds of $s(\Delta; \overline{m})$. The following is a simple generalization of a result in [Bi], [Ec] and so on:

Proposition 6.10. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n, and $\mathcal{P}$ an integral polytope decomposition of P. We assume that $\mathcal{P}$ has a lifting function. Let Δ be a convex set in $\mathbb{R}^n$ and take $P_1, \dots, P_r \in \mathcal{P}^{[n]}$ such that $\Delta_i := P_i \cap \Delta$ has the non-empty interior for each $1 \leq i \leq r$. (There may be another $P' \in \mathcal{P}^{[n]}$ such that $(P' \cap \Delta)^\circ \neq \emptyset$.) Set $s_i \in \mathbb{Z}_+$ be a positive integer and take $\overline{m}_i \in \mathbb{R}_+^{s_i}$ for each $1 \leq i \leq r$.*

Then $s(\Delta; \overline{m}_1, \dots, \overline{m}_r) \geq \min_{1 \leq i \leq r} s(\Delta_i; \overline{m}_i)$ holds.

Proof. Fix very general points $p_{i1}, \dots, p_{is_i}$ in $(\mathbb{C}^\times)^n$ for $i = 1, \dots, r$. By the existence of a lifting function, there is a toric degeneration $f : \mathcal{X} \to \mathbb{A}^1$ as above. We consider each p_{ij} as a point in the central fiber X_0 by the inclusion $(\mathbb{C}^\times)^n \subset X_{P_i} \subset X_0$. For each $1 \leq i \leq r, 1 \leq j \leq s_i$, choose a very general section σ_{ij} of f satisfying $\sigma_{ij}(0) = p_{ij}$. Set $\overline{m}_i = (m_{i1}, \dots, m_{is_i})$.

We consider the natural map

$$\varphi : V_{\bigcup_i k\Delta_i^\circ} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1} \hookrightarrow V_{kP} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1} = H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \to \bigoplus_{i,j} \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\mathcal{X}} / \mathcal{I}_{ij}^{l_{ij}+1}$$

for integers $k, l_{ij} \in \mathbb{N}$, where $\mathcal{I}_{ij} \subset \mathcal{O}_{\mathcal{X}}$ is the ideal corresponding to $\sigma_{ij}(\mathbb{A}^1)$. By similar arguments in the proofs of Proposition 5.6 or Proposition 6.6, we can show that $\varphi|_{X_t} : V_{\bigcup_i k\Delta_i^\circ} \to \bigoplus_{i,j} L_P^{\otimes k} \otimes \mathcal{O}_{X_P} / \mathfrak{m}_{\sigma_{ij}(t)}^{l_{ij}+1} = \bigoplus_{i,j} \mathcal{O}_{(\mathbb{C}^\times)^n} / \mathfrak{m}_{\sigma_{ij}(t)}^{l_{ij}+1}$ is surjective for general $t \in \mathbb{A}^1$ if so is $\varphi|_{X_0}$. Note that $V_{\bigcup_i k\Delta_i^\circ} = \bigoplus_i V_{k\Delta_i^\circ}$ and $\varphi|_{X_0} = \oplus_i \psi_i$, where

$$\psi_i : V_{k\Delta_i^\circ} \to \bigoplus_j L_{P_i}^{\otimes k} \otimes \mathcal{O}_{X_{P_i}} / \mathfrak{m}_{p_{ij}}^{l_{ij}+1} = \bigoplus_j \mathcal{O}_{(\mathbb{C}^\times)^n} / \mathfrak{m}_{p_{ij}}^{l_{ij}+1}.$$

Thus $V_{\bigcup_i k\Delta_i^\circ}$ generically separates $(l_{ij})_{i,j}$-jets if $V_{k\Delta_i^\circ}$ generically separates $(l_{ij})_j$-jets for all i. Hence

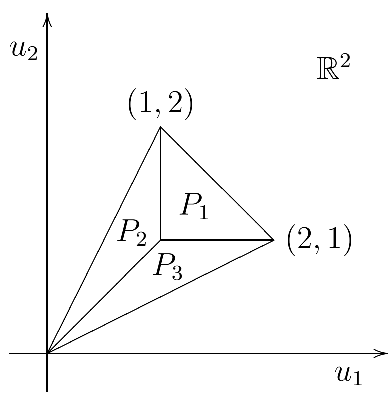
$$\begin{aligned} \tilde{s}(k\Delta; \overline{m}_1, \dots, \overline{m}_r) &= j(V_{k\Delta}; \overline{m}_1, \dots, \overline{m}_r) \\ &\geq j(V_{\bigcup_i k\Delta_i^\circ}; \overline{m}_1, \dots, \overline{m}_r) \\ &\geq \min_i j(V_{k\Delta_i^\circ}; \overline{m}_i) = \min_i \tilde{s}(k\Delta_i^\circ; \overline{m}_i) \end{aligned}$$

holds for any k. Thus we have

$$\begin{aligned} s(\Delta;\overline{m}_1,\dots,\overline{m}_r) &= \lim_k \frac{\tilde{s}(k\Delta;\overline{m}_1,\dots,\overline{m}_r)}{k} \\ &\geq \lim_k \frac{\min_i \tilde{s}(k\Delta_i^\circ;\overline{m}_i)}{k} \\ &= \min_i \lim_k \frac{\tilde{s}(k\Delta_i^\circ;\overline{m}_i)}{k} \\ &= \min_i s(\Delta_i^\circ;\overline{m}_i) \\ &= \min_i s(\Delta_i;\overline{m}_i). \end{aligned}$$

□

Example 6.11. Let $\Delta \subset \mathbb{R}^2$ be the convex hull of $(0,0),(2,1)$ and $(1,2)$. Consider the following decomposition.



It is easy to see that there exists a lifting function, so we can apply Proposition 6.10 to $P=\Delta$ and this decomposition. Then we have $s(\Delta;1,1,1) \geq \min\{s(P_1;1),s(P_2;1),s(P_3;1)\} = 1$. In this case $s(\Delta;1,1,1)=1$ holds by Lemma 4.7.

References

[An] D. Anderson, *Okounkov bodies and toric degenerations* , arXiv:1001.4566.

[BBC+] T. Bauer, C. Bocci, S. Cooper, S. Di Rocco, M. Dumnicki, B. Harbourne, K. Jabbusch, A.L. Knutsen, A. Küronya, R. Miranda, J. Roé, H. Schenck, T. Szemberg, and Z. Teitler, *Recent developments and open problems in linear series*, arXiv:1101.4363.

[BDH+] T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A.L. Knutsen, W. Syzdek, and T. Szemberg, *A primer on Seshadri constants*, Interactions of classical and numerical algebraic geometry, 33-70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[Bi] P. Biran, *Constructing new ample divisors out of old ones*, Duke Math. J. 98 (1999), no. 1, 113-135.

[BC] S. Boucksom and H. Chen, *Okounkov bodies of filtered linear series*, arXiv:0911.2923.

[De] J.P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), Lect. Notes Math. 1507, Springer-Verlag, 1992, pp. 87-104.

[Du] M. Dumnicki, *Cutting diagram method for systems of plane curves with base points*, Ann. Polon. Math. 90 (2007), no. 2, 131-143.

[Ec] T. Eckl, *An asymptotic version of Dumnicki's algorithm for linear systems in* $\mathbb{CP}^2$, Geom. Dedicata 137 (2008), 149-162.

[Ga] L.F. García, *Seshadri constants in finite subgroups of abelian surfaces*, Geom. Dedicata 127 (2007), 43-48.

[GS] M. Gross and B. Siebert, *An invitation to toric degenerations*, arXiv:0808.2749.

[KK] K. Kaveh and A. G. Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, arXiv:0904.3350.

[Kh] A. G. Khovanskii, *The Newton polytope, the Hilbert polynomial and sums of finite sets*, Funct. Anal. Appl. 26 (1992), no. 4, 276-281 (1993).

[La] R. Lazarsfeld, *Positivity in algebraic geometry I*, Springer Verlag, 2004.

[LM] R. Lazarsfeld and M. Mustaţă, *Convex bodies associated to linear series*, Ann. Sci. Ec. Norm. Super. (4) 42 (2009), no. 5, 783-835.

[MP] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. 115 (1994), no. 3, 405-434.

[Na] M. Nakamaye, *Base loci of linear series are numerically determined*, Trans. Amer. Math. Soc. 355 (2003), no. 2, 551-566.

[Ok1] A. Okounkov, *Brunn-Minkowski inequality for multiplicities*, Invent. Math. 125 (1996), no. 3, 405-411.

[Ok2] A. Okounkov, *Why would multiplicities be log-concave?*, in The orbit method in geometry and physics, Progr. Math., 213, 2003, 329-347,

[Ro] L. Robbiano, *On the theory of graded structures*, J. Symbolic Comput. 2 (1986), no. 2, 139-170.

[RT] J. Ross and R. P. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. 16 (2007), no. 2, 201-255.

Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.

E-mail address: itoatsu@ms.u-tokyo.ac.jp